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The Method of Holomorphic Foliations in Planar Periodic Systems: The Case of Riccati Equations[1]

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We develop a new method of investigation of periodic solutions of planar holomorphic polynomial differential equations with trigonometric coefficients. This [illegible]

invariants of holomorphic vector fields defining the foliations. In particular, new examples of Riccati equations without periodic solutions are presented. © 2000 Academic Press

1. INTRODUCTION

In this paper we study the qualitative properties of systems of two differential equations with periodic coefficients of the type

$$\frac{dz}{dt} = P(z, e^{it}, e^{-it}), \qquad z = z_1 + iz_2 \in \mathbf{C} \simeq \mathbf{R}^2, \tag{1}$$

where P is a polynomial in all three variables with complex coefficients. This means that P is a polynomial in z with trigonometric coefficients.

The fundamental questions concerning (1) are the existence, number, and types of its periodic solutions and the presence of chaotic behaviour. We conjecture that systems of the type (1) have either only finitely many periodic solutions or a continuous family of periodic solutions. In particular, they should not exhibit any kind of chaos.

(This conjecture is restricted only to systems with holomorphic right-hand side. Srzednicki and Wójcik [SW] have shown that the system $dz/dt = (a + e^{it}\,|z|^2)\,\bar{z}$ generates chaotic dynamics for large a.)

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The present paper is the first in the planned series devoted to confirm the above conjecture. We present a new method to study such systems. It is based on the theory of holomorphic foliations in $\mathbf{C}P^1 \times \mathbf{C}P^1$.

The effectiveness of our method can be demonstrated on the example of the equation

$$dz/dt = z^2 + re^{it}. \tag{2}$$

Srzednicki [Srz] has shown that if the parameter r is small, then (2) has at least one periodic solution. Mahwin [Maw] conjectured that there should exist an equation $dz/dt = z^2 + q(t)$, q-periodic, without periodic solutions. The first example of such an equation was given by Campos and Ortega [CO] (see also [Cam]). They used real methods. Miklaszewski [Mik] investigated the system (2) using expansion of a periodic solution into the Fourier series. He calculated approximately the value r_1 of the parameter $r > 0$ such that (2) should have no periodic solutions (the Fourier series should diverge). Unfortunately, his proof is not rigorous. Its demonstration is based on the conjecture that the series of inequalities $a_n^2 \leqslant a_{n-1} a_{n+1}$ hold; here $a_1 = 1$, $a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}$ are the coefficients of the Fourier series.

We shall show that there is an infinite sequence $0 < r_1 < r_2 < \cdots$, $r_j \to \infty$ of bifurcational values such that for any $r \neq r_j$ the system (2) has exactly one periodic solution (of period 2π) and for $r = r_j$ the system does not have any bounded periodic solution (see Proposition 4 in Subsection 5.2).

Our method of holomorphic foliations is the analytic theory of ordinary differential equations; however, not in real domain but in complex domain, where the time is also complex. In this paper we use only basic classical results from this theory: the convergence of the normalizing series in the Poincaré domain [Poi] and the analytic version of the invariant manifold theorem proved by Briot and Bouquet [BB], by Lyapunov [Ly], and by Dulac [Dul]. In future works we plan to use other results about analytic normal forms (proved in the 1980s by Martinet and Ramis and by Il'yashenko with his students) to the problem of periodic solutions of holomorphic systems (1).

I am convinced that the holomorphic foliations form the best tool to treat the system (1). The results, obtained here in the Riccati case, are complete in the sense that they show the global picture of periodic solutions; they are geometrical like the phase portraits of real planar vector fields. I was surprised why the other authors did not use these methods. Probably the reason lies in specialization. (They could not understand the previous version of the present work.) They used to their own methods, mostly infinite dimensional. I will try to convince the readers that the results presented here have some dose of beauty and it is worth to make an effort in learning analytic theory of ODEs.

The qualitative study of polynomial systems

$$dz/dt = z^n + p_1(t)\, z^{n-1} + \cdots + p_n(t), \qquad z \in \mathbf{C}$$

with 2π-periodic coefficients $p_i(t)$ was initiated by Pliss in [Pl1] and by Lloyd in [Ll1–Ll3] (see also [LN]). There a series of results about stability and bifurcations of periodic solutions were proved. In the case $n=2$ an additional analysis was performed by Hassan [Has]. In those papers the coefficients p_i are real; in the present paper we shall skip this restriction.

We cite also the recent paper by Borisovich and Marzantowicz [BM], where they consider the latter system with the coefficients $p_j(t)$ admitting Fourier expansions $\sum a_k e^{ikt}$ with only positive frequencies k. The system (1) with P not depending on e^{-it} is a special case. Using functional analytic methods they show existence of many periodic solutions. The Borisovich–Marzantowicz systems can be also treated by means of holomorphic foliations. The corresponding foliation lives in $D \times \mathbf{C}P^1$, where D is a disc.

Another application of this theory is the XVIth Hilbert problem for quadratic planar vector fields. Recall that one has to estimate the number of limit cycles of a quadratic planar autonomous vector field. It is well known that any limit cycle surrounds a singular point of the center or focus type. Therefore in suitable polar coordinates r, θ the equation for phase curves takes the form $dr/d\theta = (\lambda r + r^2 A(\theta))/(1 + rB(\theta))$, where $\lambda \in \mathbf{R}$ and A, B are cubic homogeneous trigonometric polynomials. Applying the Cherkas transformation (see [Ch1]) $\rho = r/(1 + rB)$, we arrive at the Abel equation

$$\frac{d\rho}{d\theta} = C(\theta)\,\rho + D(\theta)\,\rho^2 + E(\theta)\,\rho^3.$$

The above reduction can be also applied to systems with homogeneous nonlinearities. It is known that if E does not change sign, then this equation can have at most three 2π-periodic solutions (see [Pl1]). In [GL] it was proved that if D does not change sign, then also there are at most three periodic solutions.

In connection with the XVIth Hilbert problem, it is worth to mention the finitness theorem: any (individual) planar polynomial vector field has only finitely many limit cycles. This result, firstly formulated by Dulac (but with incomplete proof) was finally proved by Il'yashenko and by Ecalle (independently). Our conjecture about finitness of the number of periodic solutions of the polynomial system (1) would be an analogue of the finitness of the number of limit cycles. We hope that the methods developed in the proof of the latter theorem would be helpful in the non-autonomous case.

Equation (1) is a particular case of the polynomial but not holomorphic system of the type

$$dz/dt = S(z, \bar{z}, t),$$

where S is a polynomial in $z, \bar{z}$ and 2π-periodic in t, considered by Srzednicki [Srz]. Using topological methods (Ważewski principle, Lefschetz fixed point theorem, generalization of the Conley index) Srzednicki has shown existence of periodic solutions of this equation under some rather general assumptions.

Recently Manásevich, Mawhin, and Zanolin [MMZ] have generalized the results of Srzednicki to periodic complex systems

$$a_i(t) \frac{dz_i}{dt} = |z_i|^p \bar{z}_i^q + h_i(t, z_1, ..., \bar{z}_n), \qquad i = 1, ..., n.$$

Their main tools are Leray–Schauder type theorems.

In the both papers [Srz, MMZ] the considered systems consist of two parts: a dominant highest order term and a rest, which can be treated as perturbation for large amplitudes.

The possible generalizations of the method developed here for these kind of systems would be the following.

For the equation

$$dz/dt = S(z, \bar{z}, e^{it}, e^{-it})$$

one can associate certain foliation in $\mathbf{C}^3$ into holomorphic curves. One looks for intersections of these curves (real 2-dimensional surfaces) of the foliation with certain 3-dimensional real surface in $\mathbf{C}^3 \approx \mathbf{R}^6$.

For many-dimensional complex (holomorphic or not) systems one introduces a suitable polynomial vector field in many-dimensional complex space and one looks for intersections of its leaves with a suitable cylinder $|x| = 1$ or with a smaller real algebraic subvariety.

The theory of many-dimensional foliations is not well developed. Its application to the problem of periodic solutions seems to be difficult. We shall not do it here.

The plan of the paper is following. In Section 2 we describe the main notions and properties of the method of holomorphic foliations. There we do not make any restrictions onto the degree of the polynomial P; in the further sections we assume the Riccati case (deg $= 2$). The principal properties of Riccati systems are recalled in Section 3. In Section 4 we investigate the singular periodic solutions, i.e., the singularities at infinity. The main theorems are formulated and proved in Section 5.

2. HOLOMORPHIC FOLIATION ASSOCIATED WITH THE SYSTEM (1)

2.1. *Foliation* $\mathscr{F}^0$ *in* $\mathbf{C}^2$

Let us apply the substitution

$$e^{it}=x, \qquad e^{-it}=x^{-1}.$$

The integral curves of the system (1) $dz/dt=P(z, e^{it}, e^{-it})$, i.e., the graphics $(t, z(t)) \in S^1 \times \mathbf{C}$ ($t(\bmod 2\pi)$) of solutions, form phase curves of the following vector field in $S^1 \times \mathbf{C} = \{(x, z) : |x|=1\}$

$$\dot{x}=ix^{k+1}, \qquad \dot{z}=Q(x, z), \tag{3}$$

where k is the degree of e^{-it} in P,

$$Q(x, z)=x^k P(z, x, x^{-1})$$

and the dot denotes $d/d\tau$. (Here we can treat τ as the real time and equal to t.)

The degree of Q with respect to z is the same as the degree of P. We denote it by n. The degree of Q with respect to x is equal to the sum of degrees of P with respect to e^{it} and to e^{-it}.

The vector field (3) extends naturally to a polynomial vector field in $\mathbf{C}^2$. When we start to treat the new "time" τ as complex, then (3) defines a holomorphic singular foliation $\mathscr{F}^0$ of $\mathbf{C}^2$ into leaves which are of two types:

— equilibrium points,

— Riemann surfaces (locally parametrized by τ).

The integral curves of (1) form a foliation of the cylinder

$$C=\{(x, z) : |x|=1, z \in C\}=S^1 \times \mathbf{C}$$

which is obtained by intersection of the leaves of the foliation $\mathscr{F}^0$ with this cylinder. The leaves of $\mathscr{F}^0$ are 2-dimensional real surfaces and, in intersection with the cylinder C, form 1-dimensional real curves.

LEMMA 1. *If L is a (real) 2-dimensional leaf of the foliation $\mathscr{F}^0$, then its intersection with the cylinder C is transversal.*

Moreover, the curve $L \cap C$ is not vertical at any of its point.

Proof. The x-component of the vector field (3) at $(x_0, z_0) \in L \cap C$ is equal to ix_0^k, i.e., nonzero complex number. This implies that the (complex) 1-dimensional leaf L lies non-vertically in $\mathbf{C}^2$ near (x_0, z_0). L is locally defined as $z=z_0+a_1(x-x_0)+\cdots$. Thus L is projected locally

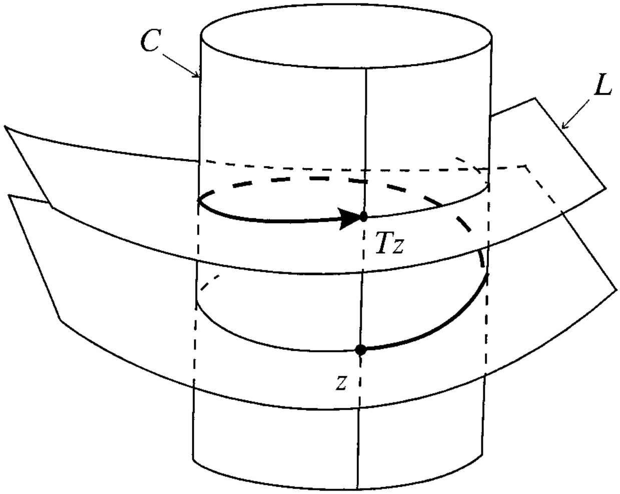

FIGURE 1

biholomorphically onto the complex x-plane (see the schematic picture in Fig. 1). From this the result of Lemma 1 follows. ∎

Lemma 1 ensures that the intersections $L \cap C$ are (real) 1-dimensional curves which are locally regularly projected onto the circle $S^1 = \{|x| = 1\}$. If

$$\Pi(x, z) = x$$

denotes this projection, then $\Pi|_{L \cap C}$ is a covering and the curve $L \cap C$ can be written locally (near $x_0 \in S^1$) in the form

$$\{(x, z(x)) : |x| = 1\}.$$

It represents a solution $z = z(t)$ of the system (1).

The case of periodic solution corresponds to the compact curve

$$\gamma = L \cap C.$$

Here the covering $\Pi|_\gamma : \gamma \to S^1$ is finite. In other cases $\Pi|_\gamma$ is infinite and defines an universal covering of S^1.

2.2. *Invariants of Periodic Leaves and the Monodromy Map*

If γ is compact, then we say that γ is *periodic* (as leaf of the foliation $\mathscr{F}^0|_C$). The rank l of the covering $\Pi|_\gamma$ (i.e., the cardinality of a fiber), multiplied by 2π, is called the *period* of γ. It is the minimal period. Lloyd says that γ is l-harmonic.

The periodic curve γ has other invariants: characteristic multiplier and multiplicity. In order to define them, we must introduce the monodromy map.

Let $\{g_s^t\}$ be the natural 2-parameter family of non-autonomous flow maps. $g_s^t(z)$ is the value at the moment t of the solution of (1) with the initial value z at the moment s.

The *monodromy map* is defined as

$$T = g_0^{2\pi} : \{1\} \times \mathbf{C} \to \{1\} \times \mathbf{C}.$$

(Here the point $1 = e^{i0}$ corresponds to $t = 0$.) Of course, the domain of definition of T is some proper subdomain of $\{1\} \times \mathbf{C} = \mathbf{C}$ but, if a point $(1, z_0)$ belongs to a $2\pi l$-periodic curve γ, then T is defined near z_0 (as a holomorphic map) and z_0 is a periodic point of the map T, $T^l(z_0) = z_0$.

The *characteristic multiplier* of the $2\pi l$-periodic curve γ is the derivative $(T^l)'(z_0)$. The characteristic multiplier can be a root of unity $(T^l)'(z_0) = e^{2\pi i p/q}$, $p, q > 0$ integers, $(p, q) = 1$ or not.

If the characteristic multiplier is not a root of unity, then the *multiplicity* of γ is equal to 1. Otherwise, the *multiplicity* of the curve γ is the multiplicity of the zero z_0 of the function $T^{lq}z - z$.

All dynamical properties of (1) are described by the dynamics of the map T.

Of course T can be defined in purely geometrical terms. The point $(1, T(z)) \in C$ is the second point in $\{x = 1\}$ of the oriented curve $\gamma_{(1, z)}$, of the foliation $\mathscr{F}^0|_C$ passing through the point $(1, z)$

2.3. *The Foliation $\mathscr{F}$ in $\mathbf{C}P^1 \times \mathbf{C}P^1$*

If $\mathscr{F}^0$ is a holomorphic foliation in $\mathbf{C} \times \mathbf{C}$, defined by a holomorphic polynomial vector field, then it can be extended to a holomorphic foliation on any algebraic compactification of $\mathbf{C} \times \mathbf{C}$. Usually the projective compactification $\mathbf{C}P^2$ is used. For the purposes of our work, the suitable compactification is $\mathbf{C}P^1 \times \mathbf{C}P^1$. We add two projective lines $\mathbf{C}P^1 \times \{\infty\}$ and $\{\infty\} \times \mathbf{C}P^1$.

In $\mathbf{C}P^1 \times \mathbf{C}P^1$ we have two projections

$$\Pi(x, z) = x, \qquad \Pi'(x, z) = z$$

and the distinguished section of "infinity" of the fibration Π

$$x \to s^\infty(x) \equiv \infty.$$

We denote its image also by s^∞ and we call it the *section at infinity*.

Let

$$C_1 = \{ |x| = 1 \} \subset \mathbf{C} \times \mathbf{C}P^1$$

be the compactification of C.

The prolongation of the foliation $\mathscr{F}^0$ from $\mathbf{C}^2$ to a neighborhoods of the two Riemann spheres at infinity is defined as follows. Consider the case of s^∞; the case $\{\infty\} \times \mathbf{C}$ is analogous. We introduce the variable

$$y = 1/z.$$

Then $s^\infty = \{y = 0\}$. The system (3), written in the variables x, y, reads as $\dot{x} = ix^{k+1}$, $\dot{y} = -y^2 P(1/y, x, x^{-1})$. If n (the degree of P in z) is greater than 2, then this vector field has pole at $y = 0$. However, we can multiply vector fields (real or complex) by functions without changing their phase portraits; only the velocities along the phase curves are changed. Multiplying the latter system by y^{n-2}, we obtain a polynomial vector field defining a holomorphic foliation near s^∞.

We denote by $\mathscr{F}$ the just defined foliation in $\mathbf{C}P^1 \times \mathbf{C}P^1$.

2.4. *Introduction of the Parameter r*

Together with the cylinder C_1 we consider the family of cylinders

$$C_r = \{ |x| = r \} \times \mathbf{C}P^1$$

$0 < r < \infty$ and the foliations

$$\mathscr{F}_r = \mathscr{F}|_{C_r}.$$

Changing the cylinder C_1 by C_r means introducing the parameter r into the system (1) as

$$\frac{dz}{dt} = P(z, re^{it}, r^{-1}e^{-it}). \tag{4}$$

Leaves of the foliation $\mathscr{F}_r$ correspond to solutions of the system (4). In particular, if $\gamma = L \cap C_1$ is a periodic leaf of the foliation $\mathscr{F}_1$, then $\gamma_r = L \cap C_r$ describes a family of solutions of Eqs. (4).

Proposition 1. *If $\gamma = L \cap C$ is periodic and r is close to* 1, *then γ_r is also periodic with the same period, characteristic multiplier and multiplicity as γ.*

Proof. The invariants of γ_r are defined by means of the monodromy maps

$$T_r\colon \{r\} \times \mathbf{C}P^1 \to \{r\} \times \mathbf{C}P^1$$

defined by means of intersections of the leaves of the foliation $\mathscr{F}|_{C_r}$ with the section $\{x=r\} \subset C_r$.

Consider the maps

$$H_r\colon \{1\} \times \mathbf{C}P^1 \to \{r\} \times \mathbf{C}P^1$$

defined by means of the lifts of the interval $[0, r]$ (in the x-plane) to the leaves of the foliation $\mathscr{F}$. It is holomorphic map, well defined near $\gamma \cap \{x=1\}$ and for r close to 1.

By construction, H_r conjugates T_1 and T_r,

$$T_r = H_r^{-1} \circ T_1 \circ H_r.$$

The result of Proposition 1 follows from the fact that the period, characteristic multiplier and multiplicity are invariants of holomorphic conjugations. ∎

2.5. *Bifurcations of Periodic Solutions*

Now we are able to explain our strategy of investigation of periodic solutions of polynomial holomorphic systems.

If a periodic solution γ lies in the open cylinder C, then its investigation is difficult. Firstly one must find it; here the authors use topological methods. Next, one must integrate the associated system in variations (in order to derive the type of γ).

We act differently. We allow γ to vary within the family γ_r and we look for its bifurcations.

Corollary to Proposition 1. *The periodic orbit γ_r does not bifurcate as long as it does not meet the section s^∞ and the* central line $C_0 = \{x=0\}$.

This means that investigation of periodic solutions of (1) is reduced to studying bifurcations of periodic curves γ_r in the following situations:

(1) when γ_r passes through s^∞ (*bifurcations at infinity*);

(2) when $r \to 0$ (*bifurcations at the central line*); and

(3) when $r \to \infty$, (this case is treated in the same way as the previous case).

The situation (1) was partly investigated by Pliss and Lloyd. They call the curves γ_{r_j}, corresponding to bifurcational values of the parameter, by the *singular periodic solutions.* Lloyd has proved that if $r \neq r_j$, then γ_r is stable in the sense that, after perturbation of the system (1), there is a periodic solution near γ_r with the same period (but maybe with smaller multiplicity). Proposition 1 gives a stronger property.

In Section 4 we study bifurcations at infinity in the Riccati case $n=2$. The general case will be investigated in another paper. The analysis of the situation near the line $x=0$ is done in Section 5.

In what follows we make the assumption that $n \leqslant 2$, i.e., we restrict the analysis to the linear and Riccati systems.

3. THE BASIC PROPERTIES OF LINEAR AND RICCATI EQUATIONS

Assume that the degree of P in (1) with respect to z is small, $n \leqslant 2$. Thus we have

$$dz/dt = A(t)\, z^2 + B(t)\, z + C(t). \tag{5}$$

This case is relatively simple and many results about it are known.

Its simplicity lies in the fact that the natural 2-parameter family $\{g_s^t\}$ of non-autonomous flow maps has simple structure.

Proposition 2. *If $n \leqslant 2$, then the maps $g_s^t\colon \mathbf{C}P^1 \to \mathbf{C}P^1$ are the Möbius maps*

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1,$$

where a, b, c, d depend on s, t. Above $a \equiv 1$, $d \equiv 1$, $c \equiv 0$ for $n=0$ and $c \equiv 0$ for $n=1$.

Proof. It follows from the fact that a general solution of the Riccati equation (5) has the form

$$z(t) = M(t)\, \frac{C_1 \dot{y}_1(t) + C_2 \dot{y}_2(t)}{C_1 y_1(t) + C_2 y_2(t)},$$

where $y_{1,2}(t)$ are two independent solutions of a second order differential equation $\ddot{y} + K(t)\, \dot{y} + L(t)\, y = 0$, associated with (5) by means of the change $z = M(t)\, \dot{y}/y$. Here K, L, M are periodic functions of t, well determined by the functions A, B, C from (5). ∎

All dynamical properties of (5) are included in the dynamics of the monodromy map $T = g_0^{2\pi}$ (and the corresponding maps T_r). So, one has to calculate only the four constants $a(0, 2\pi)$, $b(0, 2\pi)$, $c(0, 2\pi)$, $d(0, 2\pi)$ and control the solutions passing through the infinity.

The dynamics of a Möbius map is well known.

PROPOSITION 3. *Any fractional-linear map is equivalent* (*by an internal automorphism in* $PSL(2, \mathbf{C})$) *to one of the three maps*:

— *The rotation* (elliptic *map*)

$$\zeta \to e^{i\alpha}\zeta, \qquad \alpha \in \mathbf{R}.$$

If α/π *is irrational, then the map has only two periodic points* (*the fixed points* $0, \infty$) *and if* $\alpha = 2\pi p/q$, $p, q > 0$-*integers*, $(p, q) = 1$, *then all the points different from the fixed points are periodic with period* q.

— *The loxodromic* (hyperbolic) *map*

$$\zeta \to v\zeta, \qquad v \in \mathbf{C} \backslash S^1$$

with only two periodic points.

— *The translation* (parabolic) *map*

$$\zeta \to \zeta + a, \qquad a \in \mathbf{C} \backslash 0.$$

Here only one point is periodic; *it is the fixed point* (∞). *In the chart* $\xi = 1/\zeta$ *the parabolic map takes the form* $\xi \to \xi/(1 + a\xi) = \xi - a\xi^2 + \cdots$.

Remark 1. The above division of the group $PSL(2, \mathbf{C})$ is the same as the division of the group $SL(2, \mathbf{C})$ (matrices with determinant 1) into: diagonalizable with the trace in the interval $[-2, 2]$, diagonalizable with the trace outside $[-2, 2]$ and with a Jordan cell.

The structure of periodic solutions of the Riccati equation (5) was described in [Ll1]. It is the structure of periodic orbits of the monodromy map T of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C}P^1$, from which we must extract the eventual periodic orbits corresponding to solutions of (5) passing through infinity.

Moreover, from the behaviour of the foliation $\mathscr{F}$ near infinity (see the next section) it follows that the monodromy maps T and T_r are conjugated in $\overline{\mathbf{C}}$ by internal automorphisms from $PSL(2, \mathbf{C})$ (see also Proposition 2 and the proof of Proposition 1).

The following natural questions arise.

(a) Given a conjugation class in $PSL(2, \mathbf{C})$ of the monodromy map, find examples of trigonometric Riccati systems with monodromy maps T_r from this class. It would be desirable to describe as large classes of these equations as possible.

(b) Construct examples of Riccati systems with parabolic monodromy maps T_r and such that for some values r_j (of the parameter r)

the fixed point of T_r lies at infinity. In such a case we would obtain examples of Riccati systems without periodic solutions in $\mathbf{C}$.

We shall partly answer these questions.

Because the linear cases $n=0$ and $n=1$ are trivial, in the sequel we assume that $n=2$.

4. THE BIFURCATIONS AT INFINITY

Let us fix the Riccati system

$$\dot{x}=ix^{k+1}, \qquad \dot{z}=D(x)\,z^2+E(x)\,z+F(x), \tag{6}$$

where $D\not\equiv 0$, E, F are polynomials.

In order to investigate the behaviour of its phase portrait near $z=\infty$, we introduce the variable $y=1/z$. We obtain

$$\dot{x}=ix^{k+1}, \qquad \dot{y}=-D(x)-y(E+Fy). \tag{7}$$

We study the leaves L of the foliation, given by the vector field (7), which pass the line $y=0$ at points $(x_0, 0)$, where

$$x_0\neq 0,\ \infty.$$

If we have such a leaf, then we study the curves $\gamma_r=L\cap C_r=L\cap\{|x|=r\}$, where r is close to $r_0=|x_0|$. γ_r corresponds to a solution of the non-autonomous system (4). Usually we assume that γ_r is periodic. The projection $\gamma'_r=\Pi'(\gamma_r)$ onto the phase space $\overline{\mathbf{C}}=\mathbf{C}P^1$ of the system (4) describes the trajectory of the periodic solution in the Riemann sphere. γ'_r can have self-intersections.

THEOREM 1. (a) *If, in the case of the Riccati system* (7), *a periodic curve γ_{r_0} passes through a point $x=x_0$, $z=\infty$, then this bifurcation does not change any of its invariants* (*period, characteristic multiplier and multiplicity*). *These invariants are the same for $r<r_0$ as for $r>r_0$. Moreover, the curves γ_r are smooth for all r.*

(b) *If the point x_0 is not a zero of the polynomial $D(x)$ in* (7), *then the projections $\gamma'_r=\Pi'(\gamma_r)$ are smooth near $y=0$ and for r close to r_0.*

(c) *If $D(x_0)=0$, then γ'_{r_0}* (*near $y=0$*) *is an image of a real analytic map* $(\mathbf{R}, 0)\to\mathbf{C}P^1$.

Proof. Let L be the leaf of the foliation $\mathscr{F}$ passing through the point $x=x_0$, $y=0$ and containing the family γ_r of curves. It is a graphic of a function $x\to y(x)$, where the latter satisfies the equation

$$\frac{dy}{dx}=ix^{-k-1}D(x)+O(y), \qquad y(x_0)=0.$$

If $D(x_0)\neq 0$ and $x_0\neq 0, \infty$, then

$$L: y=a(x-x_0)+\cdots, \qquad a\neq 0, \infty.$$

If $D(x)=\mathrm{const}(x-x_0)^l+\cdots$, then

$$L: y=a(x-x_0)^{l+1}+\cdots.$$

We see that the projection $\Pi|_L: L\to \mathbf{C}$ onto the x-plane is regular near $x=x_0$, $y=0$. Because the curves γ_r are the preimages of the circles $|x|=r$ under this projection, $\gamma_r=(\Pi|_L)^{-1}(\{|x|=r\})$, then the point (a) of Theorem 1 follows.

We have $\gamma_r'=\Pi'\circ(\Pi|_L)^{-1}(\{|x|=r\})$. Because $\Pi|_L$ is a local biholomorphism, and $\Pi'|_L$ is a biholomorphism in the case $D(x_0)\neq 0$, then γ_r' is locally smooth. This gives the point (b) of Theorem 1.

If $D(x_0)=0$, then $\Pi'|_L$ is a degenerate analytic map. Thus γ_r' may be singular. Nevertheless, it is an analytic real curve. We have the point (c) of Theorem 1. ▮

Remark 2. The simplest singularity of the curve γ_r' is the cusp. It can be shown that, under a genericity assumption about the coefficients of the system (6), only the cusp singularities occur.

The first assumption is the following:

(i) *The function* $D(x)$ *has only simple zeros* x_j. We have $L: y=a(x-x_j)^2+\cdots$ for a leaf passing through such point $x=x_j$, $y=0$. Let $\hat{x}=\sqrt{a}\,(x-x_j)+\cdots$ be a new local variable such that $y=\hat{x}^2$ at the leaf L. The circle $|x|=|x_j|$ is given (in the parametric form) in the chart $\hat{x}$ as follows

$$\hat{x}=\alpha s+\beta s^2+\cdots, \qquad s\in(\mathbf{R},0)$$

The second assumption states that:

(ii) $\beta/\alpha\in\mathbf{C}\backslash\mathbf{R}$.

Assume that (i) and (ii) hold. We get $\gamma_{r_j}': y=\alpha^2 s^2+2\alpha\beta s^3+\cdots$. Here the complex numbers α^2 and $2\alpha\beta$ form a real basis of $\mathbf{C}$ treated as $\mathbf{R}^2$.

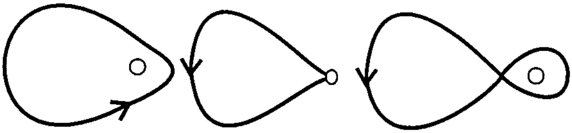

FIGURE 2

Representing $y \in \mathbf{C} \approx \mathbf{R}^2$ in this basis $\alpha^2, 2\alpha\beta$, i.e., $y = y_1 \cdot \alpha^2 + y_2 \cdot 2\alpha\beta$, we get $y_1 = s^2 + \cdots$, $y_2 = s^3 + \cdots$ or

$$y_1^3 \approx y_2^2.$$

This is the cusp.

The bifurcations of γ_r' in the case $D(x) = \text{const}(x - x_j) + \cdots$ are shown at Fig. 2.

5. THE PHASE PORTRAIT NEAR THE LINE $x = 0$

5.1. *Division into Cases*

In this section we study the behaviour of the leaves of the foliation $\mathscr{F}$ near the projective line $C_0 = \{x = 0\}$. It is natural to expand the right-hand side of the vector field (3) into powers of x.

However, before doing it, we note that the vector field (3) can be sometimes simplified. For example, if the initial non-autonomous system has the form $dz/dt = e^{i(l+1)t}\tilde{P}(z, e^{it})$, $l \geqslant 0$, $\tilde{P}$-polynomial, then the degree k of e^{it} in (1) is zero and the vector field (3) can be divided by x (without changing the phase portrait). What we obtain is the system

$$\dot{x} = i, \qquad \dot{z} = x^l \tilde{P}(z, x). \tag{8}$$

Otherwise we have

$$\dot{x} = ix^{k+1}, \qquad \dot{z} = P_0(z) + xP_1(z, x). \tag{9}$$

We distinguish three cases:

— the system (8),

— the system (9) with $k = 0$, and

— the system (9) with $k > 0$.

5.2. *The System* (8)

THEOREM 2. *Any Riccati system of the form*

$$\frac{dz}{dt} = e^{i(l+1)t}\tilde{P}(z, e^{it}), \qquad l \geqslant 0,$$

where $\tilde{P}$ is a polynomial (quadratic in z) has all its solutions 2π-periodic in the Riemann sphere. In **C** *there is only a real* 1*-parameter family of singular periodic solution (escaping to infinity) and all the other solutions are 2π-periodic.*

EXAMPLE 1. The system $dz/dt = e^{it}z^2 + e^{3it}$ has all solutions 2π-periodic in $\bar{\mathbf{C}}$.

Proof of Theorem 2. Because the x-component of the vector field (8) is nonzero, through any point $(0, z_0) \in C_0$ a smooth analytic leaf

$$L(z_0) : z = z_0 + O(x)$$

of the foliation $\mathscr{F}$ passes.

If r is small, then the intersection $L(z_0) \cap C_r = \gamma_r(z_0)$ is a closed curve: $x(t) = re^{it}$, $z(t) = z_0 + O(r)$ with 2π-periodic $z(t)$. This means that the system (3) has continuous family of periodic solutions with period 2π. The monodromy maps are equal to identity.

As r grows the periodic curves $\gamma_r(z_0)$ may grow and pass through the section s^∞. But Theorem 1 from the previous section states that this bifurcation is safe. We have $T_r \equiv id$ in $\bar{\mathbf{C}}$ and for any $r > 0$, in particular for $r = 1$. This shows the first part of Theorem 2, any solution is 2π-periodic in the Riemann sphere.

If we take into account only the solutions running through the finite part of the Riemann sphere, then we have to exclude the solutions passing through the point at infinity (singular periodic solutions). The best way to see the singular periodic solutions is to pass to the chart x, $y = 1/z$ near s^∞, like in Section 4.

From the proof of Theorem 1 we find that the local equations for the leaf through the point $x = x_0$, $y = 0$ is

$$L(x_0) : y = a(x - x_0)^k + \cdots.$$

The singular periodic solution through $x = x_0$, $y = 0$ is $\gamma(x_0) = L(x_0) \cap \{|x| = 1\}$. Thus the singular periodic solutions are parametrized by $x_0 \in S^1$. Because for different x_0's the leaves $L(x_0)$'s are different (in general), the family of singular periodic solutions is real 1-parameter family. The set of initial conditions $x=1$, $z=z_0$, laying in singular periodic solutions, forms a real 1-dimensional curve in $\mathbf{C}\simeq\mathbf{R}^2$; this curve can have self-intersections. ▮

5.3. *The System* (9) *with* $k=0$

Now we consider the system (9) with $k=0$ and $P_0(z)\not\equiv 0$. Here the line $x=0$ is invariant and the vector field (9) restricted to it is a quadratic holomorphic vector field

$$\dot{z}=P_0(z).$$

Note that this vector field is holomorphic not only in the finite part $\mathbf{C}$ but on the whole $C_0\simeq\mathbf{C}P^1$. In passing to the chart $y=1/z$ one does not need to change the time. Only Riccati vector fields have this property. (In algebraic geometry they are treated as global sections of certain holomorphic vector bundle over $\mathbf{C}P^1$, the holomorphic tangent bundle.).

The vector field $\dot{z}=P_0(z)$ has either two distinct singular points in $\mathbf{C}P^1$ or one singular point of multiplicity 2. If $P_0(z)=a(z-z_1)(z-z_2)$, $z_1\neq z_2$, then the singular points are $z_{1,2}$; if $P_0(z)=a(z-z_1)$, then the singular points are z_1 and ∞; if $P_0(z)=a(z-z_1)^2$, then the singular point z_1 has multiplicity 2; if $P_0(z)\equiv a$, then the singular point ∞ has multiplicity 2.

First we focus our attention on the situation with two distinct singular points at the line C_0. At least one of them is finite and we can assume that it is $x=z=0$, i.e., $z_1=0$. The matrix of the linear part of (9) is

$$\begin{pmatrix} i & 0 \\ * & P_0'(z_1) \end{pmatrix}$$

with the eigenvalues $\lambda_1=i$, $\lambda_2=P_0'(0)\neq 0$.

Important invariant of a singular point of a holomorphic foliation is the *ratio of eigenvalues* of the linearization of the corresponding vector field. In our case it is the ratio $\lambda(0,z_1)$ of the tangent (to C_0) eigenvalue to the normal eigenvalue and equals to $-iP_0'(z_1)$.

Lemma 2. *For the other singular point* $(0,z_2)$ *the corresponding ratio is opposite*, $\lambda(0,z_2)=-\lambda(0,z_1)$.

Remark 3. The holomorphic vector field $\dot{z}=P_0(z)$, treated as an autonomous system with real time, has interesting properties. For example, the point $(0,z_1)$ is center iff the eigenvalue $P_0'(z_1)$ is pure imaginary. This center turns out to be isochronous (with constant period function).

In order to make further progress, we have to apply some elements of the theory of analytic normal forms of analytic vector fields near singular points. We shall present only the facts needed for the purposes of our problem. For other information we refer the reader to [AI].

The first theorem concerns the case with nonzero eigenvalues

$$\dot{x}=\lambda_1 x, \qquad \dot{y}=\lambda_2 y+\cdots, \qquad \lambda_1\lambda_2\neq 0. \tag{10}$$

THEOREM 3. (a) *If the ratio* $\lambda=\lambda_2/\lambda_1$ *is not a non-negative real number, i.e.,* $\lambda\in\mathbf{C}\backslash\mathbf{R}_+\backslash 0$, *then the system* (10) *has two* analytic *invariant curves* (*separatrices*) $x=0$ *and* $\tilde{y}=y+\cdots=0$.

(b) *If* λ *is not a non-positive real number, i.e.,* $\lambda\in\mathbf{C}\backslash R_-\backslash 0$, *then there is a system of local* analytic *coordinates* x, $\tilde{y}=y+\cdots$ *such that*

$$\dot{x}=\lambda_1 x, \qquad \dot{\tilde{y}}=\lambda_2\tilde{y}+\sigma x^{\lambda} \tag{11}$$

where $\sigma=0$ *if* λ *is not a natural number.*

Here, if $\sigma=0$ *and* λ *is rational, then the system* (11) *has the family of local analytic invariant curves*

$$\tilde{y}=Cx^{\lambda};$$

if $\sigma\neq 0$, *then only the line* $x=0$ *is invariant.*

Remark 4. The first part of this theorem is the analytic analogue of the invariant manifold theorem. It was firstly proved by Briot and Bouquet in [BB]. This proof can be also found in the papers of Lyapunov [Ly] and Dulac [Dul] in the analytic case. In the finitely smooth case the invariant manifold theorem was also proved by Hadamard [Had] and by Perron [Per]. Anosov and some other authors call this theorem the Hadamard–Perron theorem. Although the result is classical, the only monograph where I have found its (not very difficult) proof in the analytic version is the book of Pliss [Pl2].

The second part of Theorem 3 forms a special case of the Poincaré theorem which says that, if the convex hull of eigenvalues (of the matrix of linear part of a vector field) is separated from 0, then the reduction of the vector field to its normal form is analytic. The proof can be found in [Ar], for example.

If $\lambda\in\mathbf{C}\backslash\mathbf{R}$, then we say that the singular point is *focus*; if $\lambda>0$, then it is *node*; if $\lambda<0$, then it is *saddle*.

If $\lambda=p/q$, where $p>0$, $q>0$ are integers and $(p,q)=1$, then the point $x=y=0$ is $(p:q)$-*resonant node*. If, additionally, the invariant $\sigma=0$ in (11), then we say that this node is *linearizable*; otherwise it is *non-linearizable*.

If $\lambda = -p/q$ with integer $p > 0$, $q > 0$, then we say that the singular point is $(p : -q)$*-resonant saddle.*

THEOREM 4. *Consider the Riccati system*

$$\frac{dz}{dt} = P(z, e^{it}), \tag{12}$$

where P is a polynomial such that $P(z, 0)$ has two distinct zeroes $z_1, z_2 \in \mathbf{C}P^1$. Assume that $\lambda = -iP'_z(0, z_0) \in \mathbf{C}\backslash\mathbf{R}_- \backslash 0$ and either λ is not a positive integer or λ is a natural number and the invariant $\sigma = 0$ in the normal form (11).

Then the monodromy maps T_r are conjugated to the maps

$$\zeta \to e^{2\pi i\lambda}\zeta.$$

Thus if λ is not rational, then the system (12) *in $\bar{\mathbf{C}}$ has exactly two periodic solutions* (*both of period* 2π); *if $\lambda = p/q$, $(p, q) = 1$ is rational, then there are two 2π-periodic solutions and all the other solutions are $2\pi q$-periodic.*

EXAMPLE 2. The system

$$dz/dt = z + z^2 + 2e^{2it}$$

has at most two periodic solutions (of period 2π). Here $\lambda(0, 0) = -i$ and T is hyperbolic.

EXAMPLE 3. The system

$$dz/dt = (4i/3)(z - z^2) + ze^{2it} + e^{3it}$$

has all solutions in $\bar{\mathbf{C}}$ periodic. Two of them are 2π-periodic and all the other solutions are 6π-periodic. Here $\lambda(0, 0) = 4/3$ and T is conjugate to the rotation by the angle $2\pi/3$.

Proof of Theorem 4. Assume that $z_1 = 0$. Let us apply Theorem 3 (the point (b)) to the system (9) near the point $(0, 0)$. We obtain that it is analytically linearizable. In local analytic coordinates $x, \tilde{z} = z + \cdots$ we have $\dot{x} = ix$, $\dot{\tilde{z}} = i\lambda\tilde{z}$.

The curve $\tilde{z} = 0$ is invariant. It forms a separatrix of the singular point. The curve $L = \{\tilde{z} = 0\}\backslash(0, 0)$ is a leaf of the foliation $\mathscr{F}$. As in the proof of Theorem 2 we find that L supports a family

$$\gamma_r = L \cap C_r, \qquad r \sim 0$$

of 2π-periodic solutions of the non-autonomous systems (4).

By Theorem 3(b), the other phase curves near $(0, 0)$ are of the form

$$\tilde{z} = Cx^{\lambda}.$$

Let us look at the monodromy maps T_r, expressed in the chart $\tilde{z}$. So, we take the path $x = re^{it}$, $0 \leqslant t \leqslant 2\pi$ in the x-plane and we lift it to a leaf $L(\tilde{z}_0)$ through the point $(r, \tilde{z}_0)$. Thus $C = \tilde{z}_0 r^{-\lambda}$ and

$$\tilde{z}(t) = e^{i\lambda t}\tilde{z}_0$$

We obtain $T_r(\tilde{z}_0) = \tilde{z}(2\pi) = e^{2\pi i\lambda}\tilde{z}_0$

The chart $\tilde{z}$ is not a projective chart; (for fixed x the map $z \to \tilde{z}$ should not be a Möbius map). However, all T_r's, expressed in the z-chart, belong to one conjugacy class in $PSL(2, \mathbf{C})$. In order to determine this class it is enough to calculate the derivative T_r' at the fixed point p_r. We take the limit $\lim_{r\to 0} T_r$. Because $\tilde{z}(x, z) \to z$ as $x, z \to 0$, we see that all T_r's are conjugate to $z \to e^{2\pi i\lambda}z$. ▮

The next case to consider is the case when the system (9) has two distinct singularities $(0, 0)$ and $(0, z_2)$, where $(0, 0)$ is a $(N:1)$-resonant node with the invariant $\sigma \neq 0$ in (11). The point $(0, z_2)$ is a resonant node.

THEOREM 5. *Let the system*

$$\frac{dz}{dt} = P(z, re^{it}) \tag{13}$$

be such that $P(z, 0)$ has two distinct zeroes z_1, z_2 and the singular point $(0, z_1)$ is $(N:1)$-resonant non-linearizable node, i.e., $\sigma \neq 0$ in (11).

Then the monodromy maps T_r, $r \neq 0$ are parabolic with one fixed point corresponding to unique periodic solution in $\overline{\mathbf{C}}$. In particular, if the leaf of the foliation $\mathscr{F}$, containing this periodic solution, intersects the section at infinity at a point (x_0, ∞), then the system (13) *with $r = |x_0|$ does not have any periodic solutions in* $\mathbf{C}$.

EXAMPLE 4. The systems

$$dz/dt = i(z + z^2) + re^{it} \tag{14}$$

have parabolic monodromy maps with unique periodic solutions $z(t) = -1 + O(r)$ for small r's. Here $\lambda(0, 0) = 1$ and $\sigma = 1$ (the linear part is the Jordan cell).

The problem of intersections of a leaf with the section at infinity will be considered later.

Proof of Theorem 5. We calculate the monodromy map T_r for small r.

LEMMA 3. *If the assumptions of Theorem* 5 *hold, then the monodromy maps* T_r, $r \to 0$ *have the following forms near* $z = 0$

$$T_r(z) = 2\pi\sigma r^N + z + \cdots$$

Proof. In the coordinate $\tilde{z} = z + \cdots$ from the point (b) of Theorem 3 ($\dot{x} = ix$, $\dot{\tilde{z}} = iN\tilde{z} + \sigma x^N$), the equations for the phase curves of the system (11) are the following $\tilde{z} = Cx^N - i\sigma x^N \ln x$, $C = \text{const}$. Thus for $x(t) = re^{it}$ we have

$$\tilde{z}(t) = Cr^N e^{iNt} - i\sigma r^N e^{iNt} \ln r + \sigma r^N e^{iNt} t$$

T_r, expressed in the chart $\tilde{z}$, have the forms

$$\tilde{z} \to \tilde{z} + 2\pi\sigma r^N$$

Passing to the chart $z = \tilde{z} + \cdots$, we obtain the result. ∎

We see that $T_r \neq id$ for $r \neq 0$ but $T_r \to id$ as $r \to 0$. On the other hand, all T_r's are mutually conjugated. This phenomenon may hold in the Möbius group $PSL(2, \mathbf{C})$ iff the maps T_r are parabolic, conjugated to $\zeta \to \zeta + a$. (Note that a is not an invariant of the conjugations, e.g., $v^{-1} \circ (id + a) \circ v = id + v^{-1}a$.)

Therefore, the maps T_r have fixed points q_r (as parabolic maps) and $T_r(z) = z + a_r(z - q_r)^2 + \cdots$ near q_r. The points q_r represent 2π-periodic curves $\delta_r \subset C_r$ of the foliation $\mathscr{F}_r$. The curves δ_r lie in one leaf L_2 of the foliation $\mathscr{F}$.

It is clear that L_2 is the (punctured) separatrix of the other singular point $(0, z_2)$, $(N : -1)$-resonant saddle. This separatrix exists and is analytic due to the point (a) of Theorem 3. ∎

Remark 5. The proofs of the last two theorems were based on local analysis near the singular point $(0, z_1)$, where $\lambda(0, z_1) \in \mathbf{C} \backslash \mathbf{R}_- \backslash 0$. One can try to repeat this analysis near the point $(0, z_2)$, which is either an (analytically linearizable) focus or a saddle. There $\lambda \in \mathbf{C} \backslash \mathbf{R}_+ \backslash 0$ and the point (a) of Theorem 3 ensures existence of invariant analytic separatrix. That separatrix supports the family of periodic curves δ_r. They define the fixed point q_r of the maps T_r.

Let $\hat{z} = z - z_2 + \cdots$ be such analytic variable that $\hat{z} = 0$ is the equation for the separatrix of $(0, z_2)$. Hence $\dot{x} = ix$, $\dot{\hat{z}} = \hat{z}(-i\lambda + \cdots)$ what implies that $d\hat{z}/dx = (\hat{z}/x)(-\lambda + \cdots)$. The calculation of T_r in the chart $\hat{z}$ gives

$$\hat{z} \to e^{-2\pi i\lambda}\hat{z} + \cdots.$$

The derivative $T_r'(q_r)$ is equal $e^{-2\pi\lambda}$; it is the reciprocal of the analogous derivatives at p_r, (if the latter exists). If $e^{2\pi\lambda} \neq 1$, then T_r is conjugated to multiplication by $e^{-2\pi\lambda}$.

However, if $e^{-2\pi\lambda} = 1$, then we cannot conclude the properties of T_r from the local behaviour of the foliation near $(0, z_2)$. Here the point $(0, z_2)$ is a $(N : -1)$-resonant saddle and one must take into account other terms of the Taylor expansion of (10). (We do not have a statement about analytic normal form, at least in the saddle case.) We did not presented these resonant terms in Theorem 3, because the dynamics is determined by the behaviour of the foliation near the $(N : 1)$-resonant node $(0, z_1)$. If $\sigma = 0$ in (11), then the resonant terms near $(0, z_2)$ are absent. If $\sigma \neq 0$, then the vector field near $(0, z_2)$ cannot be linearizable.

We consider now the case of the system (9) with a singular point at the central line C_0 of multiplicity 2. Assume that this point is $(0, 0)$.

The eigenvalues of the linearization of (9) are $\lambda_1 = i$, $\lambda_2 = 0$. It is the so-called *saddle-node*.

We need a theorem about qualitative behaviour of the foliation $\mathcal{F}$ near such singularity. In the next theorem we cite only the facts about saddle-nodes needed for our purposes. For other properties we refer the reader to [AI].

THEOREM 6. (a) *An analytic vector field*

$$\dot{x} = \lambda_1 x + \cdots, \qquad \dot{y} = \cdots, \qquad \lambda_1 \neq 0$$

can be formally (*i.e., at the level of formal power series*) *transformed to the system*

$$\dot{\tilde{x}} = \lambda_1 \tilde{x}(1 + \cdots), \qquad \dot{\tilde{y}} = \tilde{y}^{m+1}(1 + \cdots)$$

$\tilde{x} = x + \cdots$, $\tilde{y} = y + \cdots$, *where m is called the* codimension *of the saddle-node.*

(b) *There exists an analytic separatrix* $\tilde{y} = 0$, *called the* stable manifold, *corresponding to the nonzero eigenvalue. The* (*formal*) center manifold $\tilde{x} = 0$, *corresponding to the zero eigenvalue, may be not analytic.*

The formal normal form from the point (a) of this theorem is a consequence of the Poincaré-Dulac theorem (see [Ar] for the proof). The analyticity of the stable manifold was proved by Briot and Bouquet [BB] and by Dulac [Dul] in the same way as the analyticity of separatrices of a saddle was proved.

THEOREM 7. *Let the vector field* $\dot{z} = P(z, 0)$ (*in the system* (13)) *has a double singular point in* $\overline{\mathbf{C}}$. *Then the monodromy maps* T_r *are parabolic with* *a family of fixed points corresponding to a family of unique* 2π*-periodic solutions of* (13).

In particular, if the leaf of the foliation $\mathscr{F}$ *supporting these periodic solutions crosses the section at infinity at a point* (x_0, ∞), *then the system* (13) *with* $r = |x_0|$ *does not have any bounded periodic solution.*

EXAMPLE 5. The Srzednicki's system (2) has parabolic monodromy maps.

Proof of Theorem 7. We can assume that

$$\dot{x} = ix, \qquad \dot{z} = az^2 + x(b + \cdots), \quad a \neq 0.$$

We see that the center manifold is $x = 0$ and is *analytic* (it is not ensured in Theorem 6). The stable manifold is of the form

$$z = ibx + \cdots \tag{15}$$

and is *analytic* (by Theorem 6). Because the restriction of the vector field to the center manifold is $\dot{z} = z^2$, the saddle-node is of codimension 1. The formal normal form is $\dot{x} = ix$, $\dot{\tilde{z}} = \tilde{z}^2(a + \cdots)$, where $\tilde{z} = z - ibx + \cdots$ is analytic.

The stable separatrix (15) defines a leaf L with regular projection onto the x-plane. As we know, this leaf supports a family of 2π-periodic curves $\gamma_r = L \cap C_r$.

The points $p_r = \gamma_r \cap \{x = r\}$ are the fixed points of the monodromy maps T_r. To calculate the expansion of $T_r(z)$ near p_r we use the formal normal form from Theorem 6. Then $p_r = \{\tilde{z} = 0\}$ and we get

$$d\tilde{z}/dx = (\tilde{z}/x)(-ia + \cdots), \qquad x = re^{it}.$$

After simple calculations we find that

$$T_r : \tilde{z} \to \tilde{z} + 2\pi a\tilde{z}^2 + \cdots.$$

Because T_r are fractional-linear maps they are parabolic with unique fixed points p_r. ∎

Theorems 5 and 7 give indications that there may exist examples of trigonometric Riccati systems without periodic solutions. One needs only to know that the leaf $L : z = f(x)$, supporting the family of unique 2π-periodic curves γ_r, intersects the section at infinity.

Before formulating the next theorem concerning this topic we prove the following preparation lemma.

LEMMA 4. *Consider a Riccati equation* $dz/dx = a(x) z^2 + b(x) z + c(x)$ *such that for some point* $\tilde{x}$ *and constants* K, M, $R > 0$ *the following conditions hold*:

(i) $|\tilde{x}| = 3R > 0$, $|\tilde{z}| = M > 0$, $\tilde{z} = f(\tilde{x})$;

(ii) $|a(x)| > 3K$, $|b(x)| < KM$, $|c(x)| < KM^2$ *for* $|x - \tilde{x}| < R$;

(iii) $KMR > 1$.

Then there exists a (real) path $[0, s_*] \ni s \to x(s) \in \{|x - \tilde{x}| < R\}$ *such that* $x(0) = \tilde{x}$, $|dx/ds| = 1$ *and* $f(x(s_*)) = \infty$.

Proof. If $|x - \tilde{x}| < R$ and $|z| \geqslant M$, then the condition (ii) implies that

$$|dz/dx| > K\,|z|^2.$$

It is a statement about length of the vectors $dz/dx \in \mathbf{C} \simeq \mathbf{R}^2$.

Next, we will turn these vectors using an additional vector field $\zeta(x)$ in the disc $|x - \tilde{x}| < R$. The vectors $\zeta(x)$ should have unit lengths and should satisfy the property that $\partial f/\partial \zeta$ is directed in the same direction as $f(x)$. One finds

$$\zeta(x) = f(x)\,\bar{f}'(x)/|f(x)\,\bar{f}'(x)|.$$

Thus ζ is defined outside the finite set $\{ff' = 0\}$. Moreover, we have

$$\partial\,|f|/\partial\zeta = Re(f'\bar{f}\zeta)/|f| > K\,|f|^2$$

provided that $|f| \geqslant M$.

Let $x(s)$ be the integral curve with the real time s of the vector field $\zeta(x)$ with the initial condition $x(0) = \tilde{x}$. By the above

$$d\,|f|/ds > K\,|f|^2, \qquad |f|(0) = M. \tag{16}$$

Here the condition $|f| \geqslant M$ holds along the curve $\{x(s)\}$, because $d\,|f|/ds > 0$, and also the condition (ii) is preserved.

The differential inequalities in (16) imply that $|f|(s) > M/(1 - KMs)$ and for some $s = s_* < 1/(KM)$ we have $|f|(s_*) = \infty$. (Note that $M/(1 - KMs)$ satisfies the equation $\dot{x} = Kx^2$.)

By the condition (iii) the denominator $1 - KMs$ vanishes for some $s < R$; so $s_* < R$ and because $|dx/ds| = 1$, we get $|x(s_*) - \tilde{x}| < R$. ∎

Let us return to the problem of bifurcation at infinity of periodic solutions. If a leaf L is given as a graph of a polynomial, $z = a_0 + a_1 x + \cdots + a_p x^p$, then the periodic solutions corresponding to $\gamma_r = L \cap C_r$ are expanded into finite Fourier series $z(t) = a_0 + a_1 r e^{it} + \cdots + A_p r^p e^{ipt}$. Any of them cannot represent a singular periodic solution. If L is given as a graph of a rational function, (e.g., $z=1/(1-x)$), then it has only a finite number of intersections with the section at infinity.

The general result is the following.

THEOREM 8. *Assume that L is a leaf (of the foliation associated with the Riccati field (9)) laying on a separatrix $\bar{L}$ of a singular point in the central line $x=0$. Then:*

(i) *if $\bar{L}$ is not a graph of a polynomial function, then L intersects the section s^∞;*

(ii) *if $\bar{L}$ is not a graph of a rational function, then L intersects the section at infinity at infinitely many points (x_j, ∞), $|x_j|=r_j\to\infty$.*

Proof. By assumption that $\bar{L}$ is a separatrix, we have $\bar{L}$: $z=a_0+a_1x+\cdots$ near $x=0$.

(i) Assume that L does not intersect the line $z=\infty$. Then L is a graph of an integer function

$$L: z=f(x), \qquad x\in\mathbf{C}$$

The function f is a solution of the Riccati equation $dz/dx=a(x)\,z^2+b(x)\,z+c(x)$, where a, b, c are rational functions with at most one pole at $x=0$.

Because $f(x)$ is not a polynomial, for any number $N>0$ there exists a sequence of points $\hat{x}_n\to\infty$ such that $|f(\hat{x}_n)|>|\hat{x}_n|^N$.

We choose $N>\deg b-\deg a$ and $N>(1/2)(\deg c-\deg a)$ and put $\hat{z}_n=f(\hat{x}_n)$. Among the three terms $a(\hat{x}_n)\,\hat{z}_n^2$, $b(\hat{x}_n)\,\hat{z}_n$, $c(\hat{x}_n)$ the first dominates. Thus, choosing $\tilde{x}$ as one of the $\hat{x}_n$'s for large n, we get that the conditions (i) and (ii) from Lemma 4 are satisfied with the constants $K\sim|\tilde{x}|^{\deg a}$, $M\sim|\tilde{x}|^N$, $R\sim|\tilde{x}|$. We see that the third condition from Lemma 4 holds too (for large N).

By Lemma 4 the leaf L escapes to infinity for some x_* from the disc $|x-\tilde{x}|<R$. This contradicts integrity of the function f.

(ii) Assume that L is of the form $z=f(x)$, $x\neq 0$, where f is not a rational function, and that L has only finite intersection points with s^∞. The analysis performed in Section 4 shows that any point $(x_j,\infty)\in L$ represents a pole of the function f of finite order (Theorem 1). Thus there should exist a polynomial W such that the function $W(x)\,f(x)$ is integer function. Because f is not rational, Wf is not a polynomial.

Now we can simply repeat the proof from the point (i). ∎

PROPOSITION 4. (a) *For the system (2), i.e., $dz/dt=z^2+re^{it}$ (see Example 5), there exists a sequence $r_j\to\infty$ of bifurcational values such that* *for any* $r \neq r_j$ *this equation has exactly one periodic solution* (*of period* 2π) *and for* $r = r_j$, (2) *does not have any bounded periodic solution.*

(b) *The analogous statement holds for the system* (14), *i.e.*, $dz/dt = i(z + z^2) + re^{it}$ *from Example* 4.

Proof. (a) We have to show that the stable separatrix L intersects the line at infinity $z = \infty$. Let us consider the part $L_1 = \{(-u, iv) \in L : u > 0, v \in \mathbf{R}\}$ of L. L_1 is an integral curve of the real equation $dv/du = (v^2 + u)/u$ and it has infinitely many (real) branches in the domain $\{u > 0\}$. It holds because the solutions of this equation escape to infinity after finite time (represented by u). The solution tends to $v = +\infty$, then appears at $v = -\infty$, then grows and tends to $+\infty$ etc. (see Fig. 3(a)).

Another proof of existence of at least one bifurcational value r_j uses Theorem 8 and the Taylor expansion of the equation for the separatrix

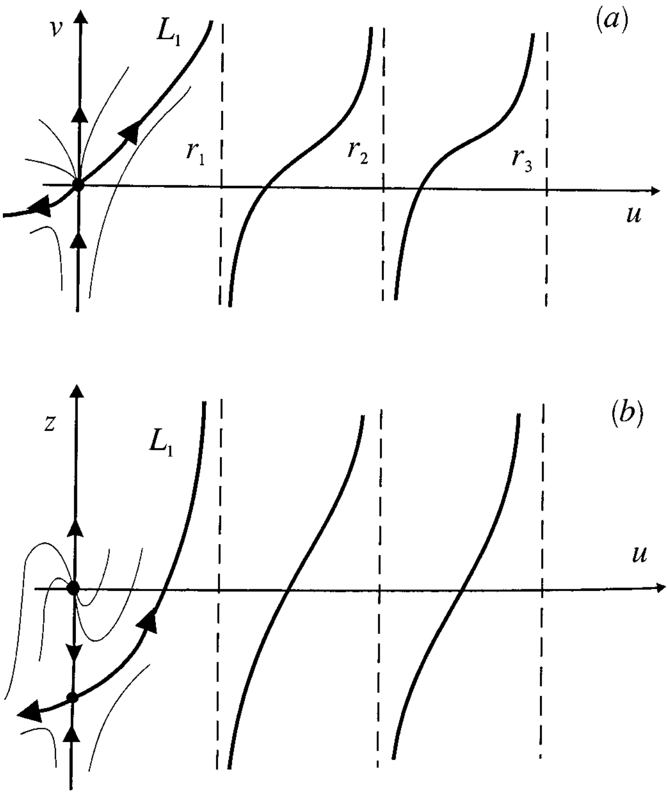

FIGURE 3

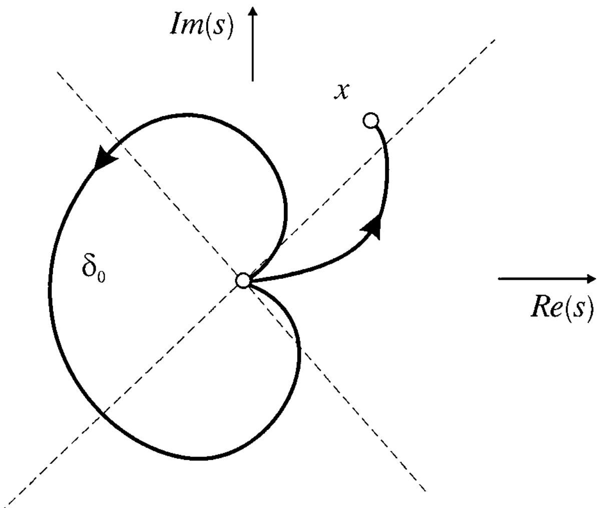

FIGURE 4

$L: z=\sum a_n x^n$, $a_1=1$, $a_n=(1/n)\sum_{k=1}^{n-1} a_k a_{n-k}>0$ (see [Mik]). Thus L is not a graph of a polynomial and must pass through s^∞.

(b) Here we put $x=-iu$, $u>0$ and $z\in\mathbf{R}$. The corresponding part L_1 of the separatrix L (of the saddle $(0,-1)$) satisfies the real equation $dz/du=(z^2+z+u)/u$ and also has infinitely many components (see Fig. 3(b)). ∎

Miklaszewski [Mik] has calculated the first bifurcational parameter $r_1\approx 1.445796...$ for the equation (2). Other values $r_2, r_3, ...$ can be computed by approximate integration of the real Riccati equation from the proof of Proposition 4.

5.4. *The System* (9) *with* $k>0$

The case of the system (9) with $k>0$ is much more difficult than the previous two cases. A detailed study will be performed in another paper. Here we consider two particular subcases: with a rational invariant curve $z=R(x)$ and with non-elementary singular point at C_0.

If the system (6) $\dot{x}=ix^{k+1}$, $\dot{z}=D(x)z^2+E(x)z+F(x)$ has an invariant algebraic curve of the form $z=R(x)$, where $R(x)$ is a rational function, then the change $z\to u=z-R(x)$ leads to the vector field with the invariant line $=0$

$$\dot{x}=ix^{k+1}, \qquad \dot{u}=D(x)u^2+E_1(x)u \tag{17}$$

Here the function D is a polynomial, equal to the old D, and $E_1=E+2DR$ is rational function (can have poles).

THEOREM 9. *Assume that the Riccati equation*

$$r^k e^{ikt} \frac{dz}{dt} = D(re^{it})\, z^2 + E(re^{it})\, z + F(re^{it})$$

has the following properties:

(i) *the equation* $ix^{k+1}\, dz/dx = D(x)\, z^2 + E(x)\, z + F(x)$ *has invariant rational curve of the form* $z = R(x)$ *and*

(ii) *the integral* $-i \int^x s^{-k-1} E_1(s)\, ds$ *has the expansion* $a_{-m} x^{-m} + a_{-m+1} x^{-m+1} + \cdots + a_{-1} x^{-1} + \alpha \ln x + \cdots$, $m > 0$ *near* $x = 0$, (*here* E_1 *is the same as in* (17)).

Then we have the following.

— *If* α *is not rational, then the system has only two periodic solutions in* $\overline{\mathbf{C}}$.

— *If* $\alpha = p/q$ *is rational and non-integer, then almost all solutions are* $2\pi q$*-periodic.*

— *If* α *is integer and the invariant*

$$A = \oint_{|w| = r} w^{-\alpha - k - 1} e^{-S(w)}\, D(w)\, dw \tag{18}$$

is nonzero, then the system has only one periodic solution.

— *If* α *is integer and* $A = 0$, *then all solutions are* 2π*-periodic.*

Proof. Introduce the variable $y = \frac{1}{u} = \frac{1}{z - R(x)}$. Then the Riccati equation takes the linear form

$$\frac{dy}{dt} = -(re^{it})^{-k}\, [E_1(re^{it})\, y + D(re^{it})].$$

The general solution of the corresponding homogeneous equation is

$$Ce^{i\alpha t} \exp S(re^{it}),$$

where $S(x) = a_{-m} x^{-m} + \cdots + a_{-2} x^{-2} + a_0 + a_1 x + \cdots$, i.e., $\alpha \ln x + S(x) = -i \int^x w^{-k-1} E_1(w)\, dw$. Thus the general solution of the non-homogeneous equation is

$$y(t) = e^{i\alpha t} e^{S(re^{it})} \left(C - \int_0^t e^{-i\alpha s} e^{-S(re^{it})} (re^{is})^{-k}\, D(re^{is})\, ds \right),$$

where C is a constant.

We investigate the monodromy mapping $T: y(0) \to y(2\pi)$. It has the affine form $y(0) \to ay(0) + b$. It is also clear that

$$a = e^{2\pi i \alpha}.$$

Thus if α is not integer, then the monodromy map is conjugated (in the group of affine automorphisms) with the linear map $y_0 \to a y_0$ and the thesis of Theorem 9 follows.

Assume that α is integer. Then, depending whether $b \neq 0$ or $b = 0$, there is only one periodic point (i.e., the fixed point at infinity) or $T \equiv id$. So, we have to calculate the difference $y(2\pi) - y(0)$. We see that it equals to

$$-r^{-k} e^{S(r)} \int_0^{2\pi} e^{-i(\alpha+k)s} e^{-S(re^{is})} D(re^{is})\, ds.$$

Up to a constant multiplier, it equals the integral in (18) defining the invariant A. ∎

Remark 6. In [Zol] it is proved that the necessary and sufficient condition to representability of the integral $\int^x w^{-\alpha-k-1} e^{-S(w)} D(w)\, dw$ in the form $x^{-\alpha} e^{-S(x)} \psi(x)$, with $\psi(x)$ holomorphic in a whole neighborhood of $x = 0$, is the vanishing of the integrals of $w^{-\alpha-k-1} e^{-S(w)} D(w)\, dw$ along the paths δ_j, $j = 0, ..., m-1$, which have the beginnings at $0e^{i(\theta_0 + 2\pi j/m)}$, the ends at $0e^{i(\theta_0 + 2\pi(j+1)/m)}$ and are fully contained in the sectors defined by their limit directions. These quantities form analogues of the Stokes operators appearing in the theory of linear meromorphic differential systems with irregular singularities.

The invariant A from Theorem 9 is the sum of these quantities.

Sometimes the invariant A can be calculated.

Example 6. For the equation

$$dz/dt = i(re^{it} + r^{-1}e^{-it})\, z + ir^d e^{idt} z^2$$

the exponent $\alpha = 0$ and the invariant A is equal to $-\int_{|x|=r} e^{x-1/x} x^{d-1}\, dx = -i \int_0^{2\pi} e^{i[d\theta - 2\sin\theta]}\, d\theta = (i/2\pi)\, J_d(2) \neq 0$, where $J_d(t)$ are the Bessel functions. We have $J_0(2) \approx 0.2238...$, $J_1(2) \approx 0.5767...$, $J_2(2) \approx 0.35283...$ (see [GM]).

Therefore, the monodromy maps T_r are parabolic.

Remark 7. The difficulty in studying the general system (9) with $k > 0$ lies in the following. Consider the system

$$\dot{x} = ix^{k+1}, \qquad \dot{z} = az + bx + \cdots, \quad a \neq 0$$

near the singular point $x=z=0$. It is the saddle-node singularity. Its stable (analytic) separatrix is $x=0$ and its (formal) center manifold is $z=-bx/a+\cdots$. However, Theorem 6 does not ensure that the formal center manifold is analytic. In fact, generally it is not analytic. When we take any leaf of the foliation $\mathscr{F}$ in the form $z=\phi(x)$, x in some sector with the vertex at 0, and try to prolong it around $x=0$, then we do not arrive at the initial leaf. The obstacles to such prolongation form the so called *Martinet–Ramis moduli* (see (AI]).

Thus the geometry of leaves near C_0 is quite complicated and we do not investigate it here.

At the end of this article we consider the case, when (9) takes the form

$$\dot{x}=ix^2, \qquad \dot{z}=az^2+x(bz+cx+\cdots), \qquad a\neq 0 \tag{19}$$

near $x=z=0$. Then $k=2$ and the vector field at C_0 is $\dot{z}=az^2$. The point $(0,0)$ is non-elementary; the both eigenvalues are equal to zero.

The natural way to study this system goes through the *blowing-up* of the singularity. We introduce the new coordinates $x, u=z/x$; (here the point $x=z=0$ is replaced by the line $x=0$ in the (x,u)-plane).

We obtain again the Riccati system

$$\dot{x}=ix, \qquad \dot{u}=[au^2+(b-i)\,u+c]+O(x). \tag{20}$$

So, the problem is reduced to investigation of the system (20) with the invariant line $x=0$ and containing only elementary singular points (at least one nonzero eigenvalue). Next, we can apply the results from Subsection 5.3.

EXAMPLE 7. The system

$$dz/dt=r^{-1}e^{-it}z^2+iz+r^3e^{3it}$$

has no periodic solutions for infinite number of values of the parameter r.

Indeed, if we put $z=re^{it}u$, then we obtain the equation $du/dt=u^2+r^2e^{2it}$.

Remark 8. The system

$$\dot{x}=ix^2, \qquad \dot{z}=az^2+x(b+\cdots), \qquad ab\neq 0$$

which seem to be simpler than (20), is more complex in fact. We do not present its analysis.

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REFERENCES

[Ar] V. I. Arnold, "Geometrical Theory of Differential Equations," Springer-Verlag, New York, 1983 (Russian, Nauka, Moscow, 1978).

[AI] V. I. Arnold and Yu. S. Il'yashenko, Ordinary differential equations, *in* "Ordinary Differential Equations and Smooth Dynamical Systems," pp. 1–148, Springer-Verlag, New York, 1997; "Fundamental'nye napravlenya," Vol. 1, pp. 7–146, VINITI, Moscow, 1985. [In Russian]

[BM] A. Borisovich and W. Marzantowicz, Multiplicity of periodic solutions for the planar polynomial equation, preprint, Un-ty of Poznań, 1998.

[BB] C. A. Briot and J. C. Bouquet, Propriétés des fonctions définies par des équations différentielles, *J. École Polytech.* **36** (1856), 133–198.

[Cam] J. Campos, Möbius transformations and periodic solutions of complex Riccati equations, preprint, 1997.

[CO] J. Campos and R. Ortega, Nonexistence of periodic solutions of a complex Riccati equation, *J. Differential Integral Equations* **9** (1996), 247–250.

[Ch1] L. A. Cherkas, Number of limit cycles of an autonomous second order system, *Differential Equations* **5** (1976), 666–668. [In Russian]

[Ch2] L. A. Cherkas, Limit cycles of autonomous systems with phase space $\mathbf{C}^2$, *Differential Equations* **23** (1987), 848–851. [In Russian]

[Dul] H. Dulac, Recherches sur les points singuliers des équations différentielles, *J. École Polytech.* **2**, No. 9 (1904), 1–125.

[GL] A. Gasull and J. Llibre, Limit cycles for a class of Abel equations, *SIAM J. Math. Anal.* **21** (1990), 1235–1244.

[GM] A. Gray and G. B. Matthews, "A Treatise on Bessel Functions and Their Applications in Physics," Macmillan, London, 1931.

[Had] J. Hadamard, Sur l'itération et les solutions asymptotiques des équations différentielles, *Bull. Soc. Math. France* **29** (1901), 224–228.

[Has] H. S. Hassan, On the set of periodic solutions of differential equations of Riccati type, *Proc. Edinburgh Math. Soc.* **27** (1984), 195–208.

[LN] A. Lins-Neto, The number of periodic solutions of the equation $\frac{dz}{dt}=\sum_{j=0}^{n} a_j(t)\, x^j$, $0 \leqslant t \leqslant 1$ for which $x(0)=x(1)$, *Invent. Math.* **59** (1980), 67–76.

[Ll1] N. G. Lloyd, The number of periodic solutions of the equations $\dot{z}=z^N+p_1(t)\, z^{N-1} + \cdots + p_N(t)$, *Proc. London Math. Soc.* **27** (1973), 667–700.

[Ll2] N. G. Lloyd, On analytic differential equations, *Proc. London Math. Soc.* **30** (1975), 430–444.

[Ll3] N. G. Lloyd, On a class of differential equations of Riccati type, *J. London Math. Soc.* **10** (1975), 1–10.

[Ly] A. M. Lyapunov, "Problème général de la stabilité du mouvement," Ann. of Math. Stud., Vol. 17, Princeton Univ. Press, Princeton, NJ, 1947 (Russian, Gostekhizdat, Moscow, 1950).

[MMZ] R. Manásevich, J. Mawhin, and F. Zanolin, Periodic solutions of complex-valued differential equations with periodic coefficients, *J. Differential Equations* **126** (1996), 355–373.

[Maw] J. Mawhin, Periodic solutions of some planar non-autonomous polynomial differential equations, *Differential Integral Equations* **7** (1994), 1055–1061.

[Mik] D. Miklaszewski, An equation $\dot{z} = z^2 + p(t)$ with no 2π-periodic solutions, *Bull. Belg. Math. Soc.* **3** (1996), 239–242.

[Per] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen, *Math. Z.* **29** (1928), 129–160.

[Pl1] V. A. Pliss, "Nonlocal Problems in the Theory of Nonlinear Oscillations," Academic Press, New York, 1966 (Russian, Nauka, Moscow/Leningrad, 1964).

[Pl2] V. A. Pliss, "Integral Sets of Periodic Systems of Differential Equations," Nauka, Moscow, 1977. [In Russian]

[Poi] H. Poincaré, "Sur les propriétés de fonctions définies par les équations différentielles partielles," Thèse, Paris, 1879.

[Srz] R. Srzednicki, On periodic solutions of planar polynomial differential equations with periodic coefficients, *J. Differential Equations* **114** (1994), 77–100.

[SW] R. Srzednicki and K. Wójcik, A geometric method for detecting chaotic dynamics, *J. Differential Equations* **135** (1997), 66–82.

[Zol] H. Żołądek, The extended monodromy group and liouvillian first integrals, *J. Dynam. Control. Systems* **4**, No. 1 (1998), 1–28.